Some Mathematical Formulas

These are a few mathematical formulas/tools which may prove useful to statisticians (and indeed anyone in a STEM-related field). It is by no means exhaustive.

1 Sums

1.1 Basic Sums

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

1.2 Geometric Sums

- $\displaystyle\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$
- $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r};\ \ |r| < 1$
- $\displaystyle\sum_{k=a}^{\infty} r^k = \frac{r^a}{1-r};\ \ |r| < 1$

1.3 Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

1.4 Taylor Series Expansion about *a*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + \dots \end{aligned}$$

1.5 Maclaurin Expansions

- $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{4}x^2 + \frac{1}{6}x^3 + \dots$
- $\displaystyle e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 1 - x + \frac{1}{4}x^2 - \frac{1}{6}x^3 + \dots$
- $\displaystyle \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2k+1} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots$
- $\displaystyle \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots$

2 Limits

2.1 Limits involving *e*

- $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$
- $\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}$

2.2 L'Hôspital's Rule

$\lim_n \frac{f(n)}{g(n)} = \lim_n \frac{f'(n)}{g'(n)}$, provided we have an indeterminate form $0/0$ or ∞/∞

3 Derivatives

3.1 Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

3.2 Quotient Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

3.3 Chain Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(g(x))\right] = f'(g(x))g'(x)$$

3.4 Derivative of an Inverse

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

3.5 Useful Derivatives

- $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x\ln(a)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x)$

4 Integrals

4.1 First Fundamental Theorem of Calculus

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{g(x)}^{h(x)} f(t)\,\mathrm{d}t\right) = f(h(x))h'(x) - f(g(x))g'(x)$$

4.2 Integration By Substitution

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

4.3 Integration By Parts

$$\int_a^b u\,\mathrm{d}v = uv\Big]_a^b - \int_a^b v\,\mathrm{d}u$$

4.4 Useful Integrals

- $\displaystyle\int x^n\,\mathrm{d}x = \begin{cases} \frac{1}{n+1}x^{n+1} & n \neq -1 \\ \ln|x| + C & n = -1 \end{cases}$
- $\displaystyle\int e^{-x}\,\mathrm{d}x = -e^{-x} + C$
- $\displaystyle\int xe^{-x}\,\mathrm{d}x = -e^{-x}(x+1) + C$
- $\displaystyle\int \frac{1}{(ax)^2 + b^2}\,\mathrm{d}x = \frac{1}{ab}\arctan\left(\frac{ax}{b}\right) + C$
- $\displaystyle\int \frac{1}{\sqrt{1-(ax)^2}}\,\mathrm{d}x = \frac{1}{a}\arcsin(ax) + C$
- $\displaystyle\int_0^\infty x^{r-1}e^{-x}\,\mathrm{d}x =: \Gamma(r)$ (called the **Gamma function**)
- $\displaystyle\int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x =: \mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ (sometimes called the **Beta function**)

5 Manipulation of Sums

- $\displaystyle\sum_{k=0}^{\infty} kp^k = \frac{p}{(1-p)^2}; \qquad |p| < 1$

  This formula is obtained by differentiating the geometric series term-by-term. (A rigorous proof would require justification of the interchange of the derivative operator and the sum, however such justification is omitted in this document). Recall that when the derivative is pulled inside the sum, the lower summation index increases by 1:

$$\begin{aligned}
\sum_{k=0}^{\infty} p^k &= \frac{1}{1-p} \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\sum_{k=0}^{\infty} p^k\right) &= \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{1-p}\right) \\
\sum_{k=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}p}(p^k) &= \frac{1}{(1-p)^2} \\
\sum_{k=1}^{\infty} kp^{k-1} &= \frac{1}{(1-p)^2} \\
\sum_{k=1}^{\infty} kp^k &= \frac{p}{(1-p)^2}
\end{aligned}$$

  The sum may easily be rewritten as beginning from $k = 0$, as $(kp^k)|_{k=0} = 0$. ■

- $\displaystyle\sum_{k=0}^{\infty} k^2 p^k = \frac{p(1+p)}{(1-p)^3}; \qquad |p| < 1$

$$\begin{aligned}
\sum_{k=0}^{\infty} kp^k &= \frac{p}{(1-p)^2} \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\sum_{k=0}^{\infty} kp^k\right) &= \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p}{(1-p)^2}\right) \\
\sum_{k=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}p}(kp^k) &= \frac{(1-p)^2 + 2p(1-p)}{(1-p)^4} \\
\sum_{k=1}^{\infty} k^2 p^{k-1} &= \frac{1-p+2p}{(1-p)^3} = \frac{1+p}{(1-p)^3} \\
\sum_{k=1}^{\infty} k^2 p^k &= \frac{p(1+p)}{(1-p)^3}
\end{aligned}$$

  The sum may easily be rewritten as beginning from $k = 0$, as $(k^2 p^k)|_{k=0} = 0$. ■